

# A nonlocal reaction diffusion equation and its relation with Fujita exponent

Shen Bian\*

Li Chen†

## Abstract

This paper is concerned with a type of nonlinear reaction-diffusion equation, which arises from the population dynamics. The equation includes a certain type reaction term  $u^\alpha(1 - \sigma \int_{\mathbb{R}^n} u^\beta dx)$  of dimension  $n \geq 1$  and  $\sigma > 0$ . An energy-methods-based proof on the existence of global solutions is presented and the qualitative behavior of solution which is decided by the choice of  $\alpha, \beta$  is exhibited. More precisely, for  $1 \leq \alpha < 1 + (1 - 2/p)\beta$ , where  $p$  is the exponent appears in Sobolev's embedding theorem defined in (3), the equation admits a unique global solution for any nonnegative initial data. Especially, in the case of  $n \geq 2$  and  $\beta = 1$ , the exponent  $\alpha < 1 + 2/n$  is exactly the well-known Fujita exponent. The global existence result obtained in this paper shows that by switching on the nonlocal effect, i.e., from  $\sigma = 0$  to  $\sigma > 0$ , the solution's behavior differs distinctly, that's, from finite time blow-up to global existence.

## 1 Introduction

In this paper, we study the following nonlocal initial boundary value problem,

$$u_t - \Delta u = u^\alpha \left( 1 - \sigma \int_{\mathbb{R}^n} u^\beta(x, t) dx \right), \quad x \in \mathbb{R}^n, t > 0, \quad (1a)$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n, \quad (1b)$$

where  $u$  is the density,  $\alpha, \beta \geq 1$ ,  $\sigma > 0$ .

---

\*Beijing University of Chemical Technology, 100029, Beijing. Email: [bianshen66@163.com](mailto:bianshen66@163.com). Partially supported by National Science Foundation of China (Grant No. 11501025), China Postdoctoral Science Foundation (Grant No. 2014M560037) and the Fundamental Research Funds for the Central Universities (Grant No. ZY1528).

†Universität Mannheim, 68131, Mannheim. Email: [chen@math.uni-mannheim.de](mailto:chen@math.uni-mannheim.de). Partially supported by the National Natural Science Foundation of China (NSFC), No. 11271218.

This model is developed to describe the population dynamics [5, 9] with the form  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u)$ , the function  $F$  is considered as the rate of the reproduction. In this paper, we will study problems with nonlocal reaction term.

As appeared in many literatures [2, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19], nonlocal type reaction terms can describe also Darwinian evolution of a structured population density or the behaviors of cancer cells with therapy. We review some of the known results on the reaction-diffusion equation with a nonlocal term. In [1], the authors considered the equation with reaction term  $F(t, u) = e^u + \int_{\Omega} e^u dx$ , where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$ , for which the above problem represents an ignition model for a compressible reactive gas, and they proved the finite time blow-up of solutions. Later on, a power-like nonlinearity  $F(t, u) = \int_{\Omega} u^p(t, y) dy - ku^q(t, x)$  was investigated by Wang and Wang [18] with  $p, q > 1$ , and they proved that solutions blow up. Moreover, [8] studied the case  $F(u) = u^p - \frac{1}{|\Omega|} \int_{\Omega} u^p(t, y) dy$ , this typical structure has mass conservation, and the authors showed that if  $p > n/(n-2)$ , the solutions will blow up in finite time with any initial data, while for  $1 < p < n/(n-2)$ , the solution exists globally. Recently, the authors in [2] studied the initial boundary value problem of (1) for  $\beta = 1$  and  $\sigma = 1$  in a bounded domain  $\Omega \subset \mathbb{R}^n$ . Global existence, uniqueness and long time behavior of solution were obtained, where the whole estimates rely on the long time asymptotics of the total mass  $\int_{\Omega} u(t, x) dx$ . However this property is not valid in the whole space case. That gives us a motivation to study the whole space case.

Another reason to study the Cauchy problem for this nonlocal reaction diffusion equation is that it has a close relation to the well-known Fujita exponent. We first list the main result of this paper, afterwards explain its relation to the Fujita exponent.

We take  $\sigma = 1$  for simplicity. All the discussions and results that obtained in this paper are valid for any positive  $\sigma$ .

**Theorem 1.** *Assume that  $u_0$  is nonnegative and  $u_0 \in L^\beta(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ ,  $n \geq 1$ . If  $\alpha$  satisfies*

$$1 \leq \alpha < 1 + (1 - 2/p)\beta, \quad (2)$$

*where  $p$  is the exponent from the Sobolev embedding theorem, i.e.*

$$\begin{cases} p = \frac{2n}{n-2}, & n \geq 3, \\ 2 < p < \infty, & n = 2, \\ p = \infty, & n = 1. \end{cases} \quad (3)$$

*then problem (1) has a unique bounded nonnegative solution. Moreover, the following a priori estimates hold true. That's for any  $t > 0$  and  $\beta \leq k \leq \infty$*

$$\int_{\mathbb{R}^n} u(t)^k dx \leq C \left( \|u_0\|_{L^\infty(\mathbb{R}^n)}, \|u_0\|_{L^\beta(\mathbb{R}^n)} \right). \quad (4)$$

Fujita in [6] showed that in the case of  $\sigma = 0$ , the Cauchy problem has no global solution for super-critical exponent  $\alpha < 1 + 2/n$  with any nonnegative initial data by using comparison principle. On the other hand, in case of  $\alpha > 1 + 2/n$ , there exists a global solution for sufficiently small initial data, and no global solutions for sufficiently large initial data. Later on, the authors in [7] considered the case  $n = 2, \alpha = 2$  and proved that it has no global solution for any nontrivial nonnegative initial data.

Compared to the case  $\sigma = 0$ , the main mathematical difficulty in studying our problem is that unlike the cases in [6, 8, 18], solutions of (1) with positive  $\sigma$  do not obey the comparison principle and mass conservation, which makes the use of many technical conditions or tools impossible. In addition, if  $1 - \sigma \int_{\mathbb{R}^n} u^\beta dx$  remains positive, it has similar structure to the case  $\sigma = 0$ , therefore our problem might have no global solution for  $\alpha < 1 + 2/n$ . However, the results obtained in theorem 1 give an opposite consequence. Furthermore, in case  $\beta = 1$ , within exactly the same range for  $\alpha$ , i.e.,  $1 < \alpha < 1 + 2/n$ ,  $\sigma = 0$  gives the finite time blow up of the solution, while  $\sigma > 0$  gives always global existence of the solution. In other words, switching on the nonlocal effect before the blow up time will prevent the solution's blow-up behavior.

Next we give a brief outline of the key estimates in order to get global existence. Actually, most of the *a priori* estimates are based on the following arguments. For any  $k \geq 1$

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + \frac{4(k-1)}{k} \int_{\mathbb{R}^n} |\nabla u^{\frac{k}{2}}|^2 dx + k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx = k \int_{\mathbb{R}^n} u^{k+\alpha-1} dx. \quad (5)$$

Because of the speciality of the exponent  $\beta + \alpha - 1$ , which means

$$\|u\|_{L^{\beta+\alpha-1}(\mathbb{R}^n)}^{\beta+\alpha-1} \leq \left( \|u\|_{L^{\beta+2(\alpha-1)}(\mathbb{R}^n)}^{\beta+2(\alpha-1)} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{1}{2}}, \quad (6)$$

we can get the estimate for  $L^{\beta+\alpha-1}$  norm from (5) in the first stage. Afterwards, the estimates for  $L^k$  norm (with finite  $k$ ) of solutions will be splitted into two cases:  $\beta \leq k \leq \beta + \alpha - 1$  and  $k > \beta + \alpha - 1$ . More precisely, the uniform boundedness of  $L^k$  norm of solutions for  $\beta \leq k \leq \beta + \alpha - 1$  is obtained by making use of the nonnegative term  $\int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx$  and the evolution of  $L^\beta$  norm, where  $L^\beta$  norm itself is an independent delicate case. While for large  $\beta + \alpha - 1 < k < \infty$ ,  $\int_{\mathbb{R}^n} |\nabla u^{\frac{k}{2}}|^2 dx$  will be applied due to Sobolev's embedding theorem. In the end, the uniform in time  $L^\infty$  norm is done by using a modified Moser type iteration argument.

## 2 Global Existence for $1 \leq \alpha < 1 + (1 - 2/p)\beta$

The main task in proving the global existence is to get the *a priori* estimates in Proposition 2. Afterwards, a direct application of the standard parabolic theory leads to global existence of a

unique solution. Before going to the proof, we need the following Sobolev inequality

**Lemma 1** ([10]). *For  $n \geq 1$ ,  $p$  is expressed by (3), let  $u \in H^1(\mathbb{R}^n)$ . Then  $u \in L^p(\mathbb{R}^n)$  and the following inequality holds:*

$$\|u\|_{L^p(\mathbb{R}^n)}^2 \leq C(n) \|\nabla u\|_{L^2(\mathbb{R}^n)}^2, \quad n \geq 3, \quad (7)$$

$$\|u\|_{L^p(\mathbb{R}^n)}^2 \leq C(n) \left( \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 \right), \quad n = 1, 2. \quad (8)$$

Here  $C(n)$  are given constants depending on  $n$ .

Based on the above Sobolev inequality, one has the following results which will be used in the proof of Proposition 2.

**Lemma 2.** *Let  $n \geq 1$ .  $p$  is expressed by (3),  $1 \leq r < q < p$  and  $\frac{q}{r} < \frac{2}{r} + 1 - \frac{2}{p}$ , then for  $v \in H^1(\mathbb{R}^n)$ , it holds*

$$\|v\|_{L^q(\mathbb{R}^n)}^q \leq C(n) C_0^{-\frac{\lambda q}{2-\lambda q}} \|v\|_{L^r(\mathbb{R}^n)}^\gamma + C_0 \|\nabla v\|_{L^2(\mathbb{R}^n)}^2, \quad n \geq 3, \quad (9)$$

$$\|v\|_{L^q(\mathbb{R}^n)}^q \leq C(n) \left( C_0^{-\frac{\lambda q}{2-\lambda q}} + C_1^{-\frac{\lambda q}{2-\lambda q}} \right) \|v\|_{L^r(\mathbb{R}^n)}^\gamma + C_0 \|\nabla v\|_{L^2(\mathbb{R}^n)}^2 + C_1 \|v\|_{L^2(\mathbb{R}^n)}^2, \quad n = 1, 2. \quad (10)$$

Here  $C(n)$  are constants depending on  $n$ ,  $C_0, C_1$  are arbitrary positive constants and

$$\lambda = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} - \frac{1}{p}} \in (0, 1), \quad \gamma = \frac{2(1-\lambda)q}{2-\lambda q} = \frac{2\left(1 - \frac{q}{p}\right)}{\frac{2-q}{r} - \frac{2}{p} + 1}. \quad (11)$$

**Proof.** First of all, it is easy to verify that  $\lambda q < 2$  if  $\frac{q}{r} < \frac{2}{r} + 1 - \frac{2}{p}$ . Therefore, by Hölder's inequality and Lemma 1 one has that for  $n \geq 3$ ,

$$\begin{aligned} \|v\|_{L^q(\mathbb{R}^n)}^q &\leq \|v\|_{L^p(\mathbb{R}^n)}^{\lambda q} \|v\|_{L^r(\mathbb{R}^n)}^{(1-\lambda)q} \\ &\leq C(n) \|\nabla v\|_{L^2(\mathbb{R}^n)}^{\lambda q} \|v\|_{L^r(\mathbb{R}^n)}^{(1-\lambda)q}. \end{aligned}$$

Since  $\lambda q < 2$ , then by Young's inequality one has

$$\|v\|_{L^q(\mathbb{R}^n)}^q \leq C(n) C_0^{-\frac{\lambda q}{2-\lambda q}} \|v\|_{L^r(\mathbb{R}^n)}^{\frac{2(1-\lambda)q}{2-\lambda q}} + C_0 \|\nabla v\|_{L^2(\mathbb{R}^n)}^2.$$

For  $n = 1, 2$ , similar proof to the case  $n \geq 3$ , by Hölder's inequality and Young's inequality we obtain

$$\begin{aligned} \|v\|_{L^q(\mathbb{R}^n)}^q &\leq \|v\|_{L^p(\mathbb{R}^n)}^{\lambda q} \|v\|_{L^r(\mathbb{R}^n)}^{(1-\lambda)q} \\ &\leq C(n) \left( \|\nabla v\|_{L^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{\lambda q}{2}} \|v\|_{L^r(\mathbb{R}^n)}^{(1-\lambda)q} \\ &\leq C(n) \left( \|\nabla v\|_{L^2(\mathbb{R}^n)}^{\lambda q} + \|v\|_{L^2(\mathbb{R}^n)}^{\lambda q} \right) \|v\|_{L^r(\mathbb{R}^n)}^{(1-\lambda)q} \\ &\leq C(n) \left( C_0^{-\frac{\lambda q}{2-\lambda q}} + C_1^{-\frac{\lambda q}{2-\lambda q}} \right) \|v\|_{L^r(\mathbb{R}^n)}^{\frac{2(1-\lambda)q}{2-\lambda q}} + C_0 \|\nabla v\|_{L^2(\mathbb{R}^n)}^2 + C_1 \|v\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Here, for simplicity,  $C(n)$  are bounded constants depending on  $n$ . This concludes the proof.  $\square$

Now we consider the a priori estimates for global existence.

**Proposition 2.** *Let  $n \geq 1$ ,  $\alpha, \beta \geq 1$ ,  $p$  is defined as in (3). Assume  $u_0$  is nonnegative and  $u_0 \in L^\beta \cap L^\infty(\mathbb{R}^n)$ . If  $\alpha$  satisfies*

$$1 \leq \alpha < 1 + (1 - 2/p)\beta, \quad (12)$$

*then any nonnegative solution of (1) satisfies that for any  $\beta \leq k \leq \infty$  and any  $t > 0$*

$$\int_{\mathbb{R}^n} u(t)^k dx \leq C(\alpha, \beta, K_0). \quad (13)$$

Here  $K_0 = \max \left\{ 1, \|u_0\|_{L^\infty(\mathbb{R}^n)}, \|u_0\|_{L^\beta(\mathbb{R}^n)} \right\}$ .

**Proof of Proposition 2.** The proof will be given step by step. Firstly, we will give the a priori estimates for  $L^k$  norm of solution for any  $k > \max \left\{ \frac{2(\alpha-1)}{p-2}, 1 \right\}$ . Then based on the a priori estimates, we will show that for any  $\beta \leq k \leq \beta + \alpha - 1$ , the  $L^k$  norm is uniformly bounded for any  $t > 0$ . Thirdly, the boundedness of  $L^k$  norm for  $\beta + \alpha - 1 < k < \infty$  is proved. Finally, it follows that the  $L^\infty$  norm of solutions is uniformly bounded by the iterative method. Thus closes the proof.

First if all, it is obtained by multiplying (1) with  $ku^{k-1}$  ( $k \geq 1$ )

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + \frac{4(k-1)}{k} \int_{\mathbb{R}^n} |\nabla u^{\frac{k}{2}}|^2 dx + k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx = k \int_{\mathbb{R}^n} u^{k+\alpha-1} dx. \quad (14)$$

Noticing the two nonnegative terms of the left hand side of (14), we firstly use the nonnegative term  $\int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx$  to control the right hand side of (14) in order to get the uniform boundedness of  $L^k$  norm of solutions for some suitable small  $k$ . For  $L^k$  norm of large  $k$ , we will take advantage of the other nonnegative term  $\int_{\mathbb{R}^n} |\nabla u^{\frac{k}{2}}|^2 dx$  to dominate the right hand side of (14). Finally, the iterative method is applied to prove the  $L^\infty$  norm of solutions.

**Step 1 (A priori estimates).** Taking  $k > \max \left\{ \frac{2(\alpha-1)}{p-2}, 1 \right\}$  and  $\max \left\{ \frac{(\alpha-1)p}{p-2}, 1 \right\} < k' < k + \alpha - 1$  such that we can let

$$v = u^{\frac{k}{2}}, \quad q = \frac{2(k + \alpha - 1)}{k}, \quad r = \frac{2k'}{k}, \quad C_0 = \frac{k-1}{k^2}, \quad C_1 = \frac{1}{2k}$$

in Lemma 2, this follows

$$\begin{aligned} \int_{\mathbb{R}^n} u^{k+\alpha-1} dx &\leq \frac{k-1}{k^2} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(k) \left\| u^{\frac{k}{2}} \right\|_{L^{\frac{2k'}{k}}(\mathbb{R}^n)}^{\frac{2b}{k}}, \quad n \geq 3, \\ \int_{\mathbb{R}^n} u^{k+\alpha-1} dx &\leq \frac{k-1}{k^2} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(k) \left\| u^{\frac{k}{2}} \right\|_{L^{\frac{2k'}{k}}(\mathbb{R}^n)}^{\frac{2b}{k}} + \frac{1}{2k} \|u\|_{L^k(\mathbb{R}^n)}^k, \quad n = 1, 2. \end{aligned} \quad (15)$$

where

$$b = \frac{(1-\lambda)(k+\alpha-1)}{1 - \frac{\lambda(k+\alpha-1)}{k}}, \quad \lambda = \frac{\frac{k}{2k'} - \frac{k}{2(k+\alpha-1)}}{\frac{k}{2k'} - \frac{1}{p}} \in (0, 1). \quad (16)$$

Thus, (14) with (15) yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \|\nabla u^{\frac{k}{2}}\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq C(k) \|u\|_{L^{k'}(\mathbb{R}^n)}^b, \quad n \geq 3. \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \|\nabla u^{\frac{k}{2}}\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq C(k) \|u\|_{L^{k'}(\mathbb{R}^n)}^b + \frac{1}{2} \|u\|_{L^k(\mathbb{R}^n)}^k, \quad n = 1, 2, \end{aligned} \quad (18)$$

Besides, with the help of interpolation inequality, in case of  $\max\left\{\frac{p(\alpha-1)}{p-2}, \beta\right\} < k' < k + \alpha - 1$ ,

$$\begin{aligned} \|u\|_{L^{k'}(\mathbb{R}^n)}^b & \leq \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{b\theta} \|u\|_{L^\beta(\mathbb{R}^n)}^{(1-\theta)b} \\ & = \left( \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{b\theta}{k+\alpha-1}} \|u\|_{L^\beta(\mathbb{R}^n)}^{b(1-\theta-\frac{\theta\beta}{k+\alpha-1})} \end{aligned} \quad (19)$$

where

$$\theta = \frac{\frac{1}{\beta} - \frac{1}{k'}}{\frac{1}{\beta} - \frac{1}{k+\alpha-1}} \in (0, 1). \quad (20)$$

In addition, simple computations show that

$$\frac{b\theta}{k+\alpha-1} < 1 \quad (21)$$

if and only if

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right) \beta. \quad (22)$$

Now we can take  $k' = \frac{k+\alpha-1+\beta}{2} \in (\beta, k+\alpha-1)$  so that

$$1 - \theta - \frac{\theta\beta}{k+\alpha-1} = 0. \quad (23)$$

From (19) and (21), using Young's inequality one obtains

$$\begin{aligned} C(k) \|u\|_{L^{k'}(\mathbb{R}^n)}^b & \leq C(k) \left( \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{b\theta}{k+\alpha-1}} \\ & \leq \frac{k}{4} \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta + C_0(k). \end{aligned} \quad (24)$$

Therefore, together with (17) and (18) one has that for any  $k > \max \left\{ \frac{2(\alpha-1)}{p-2}, 1 \right\}$  with  $p$  is showed as in (3),

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + \frac{3k}{4} \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 \leq C_0(k), \quad n \geq 3. \\ & \frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + \frac{3k}{4} \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{k+\alpha-1} dx + \frac{3(k-1)}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq C_0(k) + \frac{1}{2} \|u\|_{L^k(\mathbb{R}^n)}^k, \quad n = 1, 2. \end{aligned} \quad (25)$$

**Step 2 ( $L^k$  estimates for  $\beta \leq k \leq \beta + \alpha - 1$ ).** Based on the above a priori estimates, we firstly use the nonnegative term  $\int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{\beta+\alpha-1} dx$  to get the uniform boundedness of  $L^{\beta+\alpha-1}$  norm, as a consequence, it follows the estimates of  $L^\beta$  norm.

First of all, by Hölder's inequality one has

$$\|u\|_{L^{\beta+\alpha-1}(\mathbb{R}^n)}^{\beta+\alpha-1} \leq \left( \|u\|_{L^{\beta+2(\alpha-1)}(\mathbb{R}^n)}^{\beta+2(\alpha-1)} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{1}{2}}. \quad (26)$$

Then by virtue of Young's inequality one has

$$\|u\|_{L^{\beta+\alpha-1}(\mathbb{R}^n)}^{\beta+\alpha-1} \leq \frac{\beta + \alpha - 1}{2} \|u\|_{L^{\beta+2(\alpha-1)}(\mathbb{R}^n)}^{\beta+2(\alpha-1)} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta + \frac{1}{2(\beta + \alpha - 1)}. \quad (27)$$

From (22) and  $p > 2$  we know  $\beta + \alpha - 1 > \max \left\{ \frac{2(\alpha-1)}{p-2}, 1 \right\}$ , and then letting  $k = \beta + \alpha - 1$  in (25) one has that for  $n \geq 1$

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^{\beta+\alpha-1} dx + \int_{\mathbb{R}^n} u^{\alpha+\beta-1} dx \leq C(\beta + \alpha - 1), \quad (28)$$

which assures the following uniform estimate in time

$$\int_{\mathbb{R}^n} u^{\beta+\alpha-1} dx \leq \|u_0\|_{L^{\alpha+\beta-1}(\mathbb{R}^n)}^{\alpha+\beta-1} e^{-t} + C(\alpha, \beta). \quad (29)$$

Next, we will apply the boundedness of  $\int_{\mathbb{R}^n} u^{\beta+\alpha-1} dx$  norm to show that the  $L^\beta$  norm is also uniformly bounded in time. By taking  $k = \beta$  in (14) one obtains

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^\beta dx \leq \beta \int_{\mathbb{R}^n} u^{\beta+\alpha-1} dx \left( 1 - \int_{\mathbb{R}^n} u^\beta dx \right). \quad (30)$$

If  $\int_{\mathbb{R}^n} u_0^\beta dx \leq 1$ , then either  $\int_{\mathbb{R}^n} u(t)^\beta dx \leq 1$  for all  $t > 0$  or there exists a time interval  $[t_0, t_0 + \varepsilon)$  such that  $\int_{\mathbb{R}^n} u(t_0)^\beta dx = 1$  and  $\int_{\mathbb{R}^n} u(t)^\beta dx > 1$  and increases for  $t_0 \leq t < t_0 + \varepsilon$ . On the other hand,

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^\beta dx < 0, \quad t_0 < t < t_0 + \varepsilon,$$

which is a contradiction with the increasing of  $\int_{\mathbb{R}^n} u(t)^\beta dx$  within  $t_0 \leq t < t_0 + \varepsilon$ . Therefore,  $\int_{\mathbb{R}^n} u(t)^\beta dx \leq 1$  for any  $t \geq 0$ .

For  $\int_{\mathbb{R}^n} u_0^\beta dx > 1$ , if  $\int_{\mathbb{R}^n} u(t)^\beta dx > 1$  for all  $t > 0$ , then  $\frac{d}{dt} \int_{\mathbb{R}^n} u^\beta dx < 0$  and thus  $\int_{\mathbb{R}^n} u^\beta dx < \int_{\mathbb{R}^n} u_0^\beta dx$ . Otherwise, denote  $t_0$  to be the first time such that  $\int_{\mathbb{R}^n} u(t_0)^\beta dx = 1$ , then using the above arguments for  $\int_{\mathbb{R}^n} u_0^\beta dx \leq 1$  we know that

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^\beta dx < 0, \quad 0 \leq t < t_0, \quad \text{and} \quad \int_{\mathbb{R}^n} u^\beta dx \leq 1, \quad t \geq t_0.$$

Collecting the two cases we obtain

$$\int_{\mathbb{R}^n} u^\beta dx \leq \max \left\{ \int_{\mathbb{R}^n} u_0^\beta dx, 1 \right\}. \quad (31)$$

Hence, Hölder's inequality gives that for any  $\beta \leq k \leq \beta + \alpha - 1$ , we have

$$\int_{\mathbb{R}^n} u^k dx \leq C(k, u_0). \quad (32)$$

**Step 3 ( $L^k$  estimates for  $\beta + \alpha - 1 < k < \infty$ ).** In this step, based on (14), the nonnegative term  $\|\nabla u^{\frac{k}{2}}\|_{L^2(\mathbb{R}^n)}^2$  will be taken into account to obtain the boundedness of  $L^k$  norm for  $k > \beta + \alpha - 1$ . Letting

$$v = u^{\frac{k}{2}}, \quad q = 2, \quad r = 1 < m < 2, \quad C_0 = \frac{k-1}{k}, \quad C_1 = \frac{1}{2}$$

in Lemma 2 one has

$$\begin{aligned} \int_{\mathbb{R}^n} u^k dx &\leq \frac{k-1}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(n, k) \|u\|_{L^{k_1}(\mathbb{R}^n)}^k, \quad n \geq 3, \\ \int_{\mathbb{R}^n} u^k dx &\leq \frac{k-1}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|u\|_{L^k(\mathbb{R}^n)}^k + C(n, k) \|u\|_{L^{k_1}(\mathbb{R}^n)}^k, \quad n = 1, 2, \end{aligned} \quad (33)$$

where  $k_1 = \frac{km}{2} < k$ . We can unify that for  $n \geq 1$ ,

$$\int_{\mathbb{R}^n} u^k dx \leq \frac{2(k-1)}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(n, k) \|u\|_{L^{k_1}(\mathbb{R}^n)}^k. \quad (34)$$

On the other hand, by Hölder's inequality, for  $k > \beta + \alpha - 1$ , we can take  $k_1 = \frac{\beta+k+\alpha-1}{2} < k$  so that

$$\|u\|_{L^{k_1}(\mathbb{R}^n)}^k \leq \left( \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{k}{\beta+k+\alpha-1}}. \quad (35)$$

Hence it follows from Young's inequality that

$$\begin{aligned} \frac{3}{2} \int_{\mathbb{R}^n} u^k dx &\leq \frac{3(k-1)}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(n, k) \left( \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta \right)^{\frac{k}{\beta+k+\alpha-1}} \\ &\leq \frac{3(k-1)}{k} \left\| \nabla u^{\frac{k}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{3k}{4} \|u\|_{L^{k+\alpha-1}(\mathbb{R}^n)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^n)}^\beta + C(n, k). \end{aligned} \quad (36)$$



Substituting (36) into (25) we can have that for  $n \geq 1$  and  $k > \beta + \alpha - 1$

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^k dx + \int_{\mathbb{R}^n} u^k dx \leq C(n, k). \quad (37)$$

Simple computations arrives that for any  $k > \beta + \alpha - 1$ ,

$$\int_{\mathbb{R}^n} u^k dx \leq \|u_0\|_{L^k(\mathbb{R}^n)}^k e^{-t} + C(n, k). \quad (38)$$

**Step 4 (The  $L^\infty$  estimates).** On account of the above arguments, our last task is to give the uniform boundedness of solutions for any  $t > 0$ .

Denote  $q_k = 2^k + \beta + \alpha - 1$ , by taking  $k = q_k$  in (14), we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^{q_k} dx + \frac{4(q_k - 1)}{q_k} \int_{\mathbb{R}^n} |\nabla u^{\frac{q_k}{2}}|^2 dx + q_k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{q_k + \alpha - 1} dx = q_k \int_{\mathbb{R}^n} u^{q_k + \alpha - 1} dx. \quad (39)$$

Armed with Lemma 2, letting

$$v = u^{\frac{q_k}{2}}, \quad q = \frac{2(q_k + \alpha - 1)}{q_k}, \quad r = \frac{2q_{k-1}}{q_k}, \quad C_0 = \frac{1}{2q_k},$$

one has that for  $n \geq 1$ ,

$$\|u\|_{L^{q_k + \alpha - 1}(\mathbb{R}^n)}^{q_k + \alpha - 1} \leq C(n) C_0^{\frac{-1}{\delta_1 - 1}} \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^{\gamma_1} + \frac{1}{2q_k} \|\nabla u^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2q_k} \|u\|_{L^{q_k}(\mathbb{R}^n)}^{q_k}, \quad (40)$$

where

$$\begin{aligned} \gamma_1 &= 1 + \frac{q_k + \alpha - 1 - q_{k-1}}{q_{k-1} - \frac{p(\alpha-1)}{p-2}} \leq 2, \text{ iff } \alpha \leq 1 + \left(1 - \frac{2}{p}\right)\beta, \\ \delta_1 &= \frac{q_k - 2q_{k-1}/p}{q_k + \alpha - 1 - q_{k-1}} = O(1). \end{aligned}$$

Substituting it into (39) and with notice that  $\frac{4(q_k - 1)}{q_k} \geq 2$ , it follows

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^n} u^{q_k} dx + \frac{3}{2} \int_{\mathbb{R}^n} |\nabla u^{\frac{q_k}{2}}|^2 dx + q_k \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{q_k + \alpha - 1} dx \\ & \leq C(n) q_k^{\frac{\delta_1}{\delta_1 - 1}} \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^{\gamma_1} + \frac{1}{2} \|u\|_{L^{q_k}(\mathbb{R}^n)}^{q_k}. \end{aligned} \quad (41)$$

Applying lemma 2 with

$$v = u^{\frac{q_k}{2}}, \quad q = 2, \quad r = \frac{2q_{k-1}}{q_k}$$

and using Young's inequality, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^n} u^{q_k} dx \leq C(n) \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^{\gamma_2} + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u^{\frac{q_k}{2}}|^2 dx \\ & \leq \frac{\beta + \alpha - 1}{3} \int_{\mathbb{R}^n} u^\beta dx \int_{\mathbb{R}^n} u^{q_k + \alpha - 1} dx + C(n, \alpha, \beta) + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u^{\frac{q_k}{2}}|^2 dx. \end{aligned} \quad (42)$$

where we have used

$$\gamma_2 = 1 + \frac{q_k - q_{k-1}}{q_{k-1}} < 2, \quad q_{k-1} = \frac{q_k + \beta + \alpha - 1}{2}.$$

By summing up (41) and (42), with the fact that  $\gamma_1 \leq 2$ , we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} u^{q_k} dx + \int_{\mathbb{R}^n} u^{q_k} dx &\leq C(\delta_1) q_k^{\frac{\delta_1}{\delta_1-1}} \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^{\gamma_1} + C(n, \alpha, \beta) \\ &\leq \max[C(\delta_1), C(n, \alpha, \beta)] q_k^{\frac{\delta_1}{\delta_1-1}} \left[ \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^{\gamma_1} + 1 \right] \\ &\leq 2 \max[C(\delta_1), C(n, \alpha, \beta)] q_k^{\frac{\delta_1}{\delta_1-1}} \max \left\{ \left( \int_{\mathbb{R}^n} u^{q_{k-1}} dx \right)^2, 1 \right\}. \end{aligned}$$

Let  $K_0 = \max \left\{ 1, \|u_0\|_{L^\beta(\mathbb{R}^n)}, \|u_0\|_{L^\infty(\mathbb{R}^n)} \right\}$ , we have the following inequality for initial data

$$\int_{\mathbb{R}^n} u_0^{q_k} dx \leq \left( \max \left\{ \|u_0\|_{L^\beta(\mathbb{R}^n)}, \|u_0\|_{L^\infty(\mathbb{R}^n)} \right\} \right)^{q_k} \leq K_0^{q_k}. \quad (43)$$

Let  $d_0 = \frac{\delta_1}{\delta_1-1}$ , it is easy to see that  $q_k^{d_0} = [2^k + \beta + \alpha - 1]^{d_0} \leq [2^k + 2^k(\beta + \alpha - 1)]^{d_0}$ . By taking  $\bar{a} = 2 \max\{C(\delta_1), C(n, \alpha, \beta)\}(\beta + \alpha)^{d_0}$  in the lemma 4.1 of [4], we obtain

$$\int_{\mathbb{R}^n} u^{q_k} dx \leq (2\bar{a})^{2^k-1} 2^{d_0(2^{k+1}-k-2)} \max \left\{ \sup_{t \geq 0} \left( \int_{\mathbb{R}^n} u(t)^{q_0} dx \right)^{2^k}, K_0^{q_k} \right\}. \quad (44)$$

Since  $q_k = 2^k + \beta + \alpha - 1$  and taking the power  $\frac{1}{q_k}$  to both sides of (44), then the boundedness of the solution  $u(x, t)$  is obtained by passing to the limit  $k \rightarrow \infty$

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq 2\bar{a} 2^{2d_0} \max \left\{ \sup_{t \geq 0} \int_{\mathbb{R}^n} u(t)^{q_0} dx, K_0 \right\}. \quad (45)$$

On the other hand, by (38) with  $q_0 > \beta + \alpha - 1$ , we know

$$\int_{\mathbb{R}^n} u(t)^{q_0} dx = \int_{\mathbb{R}^n} u(t)^{\beta+\alpha} dx \leq \|u_0\|_{L^{\beta+\alpha}(\mathbb{R}^n)}^{\beta+\alpha} + C(\alpha, \beta) \leq K_0^{\beta+\alpha} + C(\alpha, \beta).$$

Therefore we finally have

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\alpha, \beta, K_0). \quad (46)$$

thus closes the proof.  $\square$

We now have necessary *a priori* estimates for the existence of global classical solutions and we know that  $u$  is uniformly bounded for any  $t \geq 0$ . Moreover, the reaction term  $u^\alpha \left(1 - \int_{\mathbb{R}^n} u^\beta dx\right)$  is bounded from below and above. Hence the uniqueness and global existence of classical solution is followed by the standard parabolic theory. This completes the proof of Theorem 1.

### 3 Conclusions

This paper proves the global existence and uniqueness of solution to Cauchy problem (1). Especially when  $\beta = 1$ , for  $1 \leq \alpha < 1 + 2/n$  with  $n \geq 2$ , it has been proved that  $u^\alpha(1 - \sigma \int_{\mathbb{R}^n} u dx)$  is bounded from below and above, therefore, if  $u^\alpha(1 - \sigma \int_{\mathbb{R}^n} u dx)$  is positive, the structure is similar to Fujita equation  $u_t = \Delta u + u^\alpha$  in the whole space. However, our equation has global solution for  $1 < \alpha < 1 + 2/n$  when  $n \geq 2$ , which is opposite to the result of Fujita equation, where no global solution exists for any initial data. The difference arises from the nonlocal term  $1 - \sigma \int_{\mathbb{R}^n} u dx$ . In the modeling of population dynamics, there are more and more nonlocal reaction diffusion equations which have been derived. However, the corresponding mathematical theory is far from complete. This paper gives a first step analysis in studying what have the nonlocal effects brought into the problem.

### References

- [1] J. W. Bebernes and A. Bressan, Thermal behaviour for a confined reactive gas, *J. Differential Equations*, **44** (1982), 118-133.
- [2] S. Bian, L. Chen, E. Latos, Global existence and asymptotic behavior of solutions to a nonlocal Fisher-KPP type problem, arXiv:1508.00063.
- [3] S. Bian and J.-G. Liu, Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent  $m > 0$ , *Comm. Math. Phys.*, **323** (2013), 1017-1070.
- [4] S. Bian, J.-G. Liu and C. Zou, Ultra-contractivity for Keller-Segel model with diffusion exponent  $m > 1 - 2/d$ , *Kinetic and Related Models*, **7**(1) (2014), 9-28.
- [5] R.A. Fisher, The wave of advance of advantageous genes, *Ann Eugenics.*, **7** (1937), 355-69.
- [6] H. Fujita, On the blowing up of solutions of the Cauchy problem for  $u_t = \Delta u + u^{1+\alpha}$ , *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **13** (1966), 109-124.
- [7] K. Hayakawa, On Nonexistence of global Solution of some semilinear parabolic differential equations, *Proc. Japan Acad.*, **49** (1973), 503-505.
- [8] B. Hu and H.M. Yin, Semilinear parabolic equations with prescribed energy, *Rend. Circ. Mat. Palermo*, **44** (1995), 479-505.

- [9] AN. Kolmogorov, IG. Petrovsky, NS. Piskunov, Investigation of the equation of diffusion combined with increasing of the substance and its application to a biology problem, Bull Moscow State Univ Ser A: Math and Mech., **6**(1) (1937), 1-25.
- [10] E.H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics. V. 14, American Mathematical Society Providence, Rhode Island, 2nd edition, 2001.
- [11] Q. Liu, Y. Chen, S. Lu, Uniform blow-up profiles for nonlinear and nonlocal reaction-diffusion equations, Nonlinear Analysis, **71** (2009), 1572-1583.
- [12] A. Lorz, S. Mirrahimi and B. Perthame, Dirac mass dynamics in multidimensional nonlocal parabolic equations, Commun. Part. Diff. Eq., **36**(6) (2011), 1071-1098.
- [13] A. Lorz, T. Lorenzi, J. Clairambault, A. Escargueil, B. Perthame, Effects of space structure and combination therapies on phenotypic heterogeneity and drug resistance in solid tumors, Bulletin of Mathematical Biology, **77**(1) (2013), 1-22.
- [14] V. Volpert, Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011.
- [15] V. Volpert, Elliptic partial differential equations. Volume 2. Reaction-diffusion equations. Birkhäuser, 2014.
- [16] V. Volpert, V. Vougalter, Existence of stationary pulses for nonlocal reaction-diffusion equations, Documenta Math., **19** (2014), 1141-1153.
- [17] X. Wang, W. Wo, Long time behavior of solutions for a scalar nonlocal reaction-diffusion equation. Arch. Math., **96** (2011), 483-490.
- [18] M. Wang and Y. Wang, Properties of positive solutions for non-local reaction-diffusion problems, Math. Methods Appl. Sci., **19** (1996), 1141-1156.
- [19] Y. Yin, Quenching for solutions of some parabolic equations with singular nonlocal terms, Dynam. Systems Appl., **5** (1996), 19-30.